

THE AUTOMORPHISMS GROUP OF $\overline{M}_{g,n}$

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ABSTRACT. Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack parametrizing Deligne-Mumford stable n -pointed genus g curves and let $\overline{M}_{g,n}$ be its coarse moduli space: the Deligne-Mumford compactification of the moduli space of n -pointed genus g smooth curves. We prove that the automorphisms groups of $\overline{\mathcal{M}}_{g,n}$ and $\overline{M}_{g,n}$ are isomorphic to the symmetric group on n elements S_n for any g, n such that $2g - 2 + n \geq 3$, and compute the remaining cases.

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INTRODUCTION

The search for an object parametrizing n -pointed genus g smooth curves is a very classical problem in algebraic geometry. In [DM] *P. Deligne* and *D. Mumford* proved that there exists an irreducible scheme $M_{g,n}$ coarsely representing the moduli functor of n -pointed genus g smooth curves. Furthermore they provided a compactification $\overline{M}_{g,n}$ of $M_{g,n}$ adding Deligne-Mumford stable curves as boundary points and pointed out that the obstructions to representing the moduli functor of Deligne-Mumford stable curves in the category of schemes came from automorphisms of the curves. However this moduli functor can be represented in the category of algebraic stacks, indeed there exists a smooth Deligne-Mumford algebraic stack $\overline{\mathcal{M}}_{g,n}$ parametrizing Deligne-Mumford stable curves. The stack $\overline{\mathcal{M}}_{g,n}$ and its coarse moduli space $\overline{M}_{g,n}$ from several decades are among the most studied objects in algebraic geometry, despite this many natural questions about their biregular and birational geometry remain unanswered. In particular we are interested in the following issue:

Question. *What are the automorphisms groups of $\overline{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$?*

The biregular automorphisms of the moduli space $M_{g,n}$ of n -pointed genus g -stable curves and of its Deligne-Mumford compactification $\overline{M}_{g,n}$ has been studied in a series of papers, for instance [BM1] and [Ro].

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Recently, in [BM1] and [BM2], *A. Bruno* and *M. Mella* studied the fibrations of $\overline{\mathcal{M}}_{0,n}$ using its description as the closure of the subscheme of the Hilbert scheme parametrizing rational normal curves passing through n points in linearly general position in \mathbb{P}^{n-2} given by *M. Kapranov* in [Ka]. It was expected that the only possible biregular automorphisms of $\overline{\mathcal{M}}_{0,n}$ were the ones associated to a permutation of the markings. Indeed *Bruno* and *Mella* as a consequence of their theorem on fibrations derive that the automorphisms group of $\overline{\mathcal{M}}_{0,n}$ is the symmetric group S_n for any $n \geq 5$ [BM2, Theorem 4.3].

The aim of this work is to extend [BM2, Theorem 4.3] to arbitrary values of g, n and to the stack $\overline{\mathcal{M}}_{g,n}$. Our main result can be stated as follows.

Theorem. *Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack parametrizing Deligne-Mumford stable n -pointed genus g curves, and let $\overline{\mathcal{M}}_{g,n}$ be its coarse moduli space. If $2g - 2 + n \geq 3$ then*

$$\mathrm{Aut}(\overline{\mathcal{M}}_{g,n}) \cong \mathrm{Aut}(\overline{\mathcal{M}}_{g,n}) \cong S_n$$

the symmetric group on n elements. For $2g - 2 + n < 3$ we have the following special behavior:

- $\mathrm{Aut}(\overline{\mathcal{M}}_{1,2}) \cong (\mathbb{C}^*)^2$ while $\mathrm{Aut}(\overline{\mathcal{M}}_{1,2})$ is trivial,
- $\mathrm{Aut}(\overline{\mathcal{M}}_{0,4}) \cong \mathrm{Aut}(\overline{\mathcal{M}}_{0,4}) \cong \mathrm{Aut}(\overline{\mathcal{M}}_{1,1}) \cong \mathrm{PGL}(2)$ while $\mathrm{Aut}(\overline{\mathcal{M}}_{1,1}) \cong \mathbb{C}^*$,
- $\mathrm{Aut}(\overline{\mathcal{M}}_g)$ and $\mathrm{Aut}(\overline{\mathcal{M}}_g)$ are trivial for any $g \geq 2$.

These issues have been investigated in the Teichmüller-theoretic literature on the automorphisms of moduli spaces $M_{g,n}$ developed in a series of papers by *H.L. Royden*, *C. J. Earle*, *I. Kra*, *M. Korkmaz*, and others, [Ro], [EK] [Ko]. A fundamental result, proved by *Royden* in [Ro], states that the moduli space $M_{g,n}^{un}$ of genus g smooth curve marked by n unordered points has no non-trivial automorphisms if $2g - 2 + n \geq 3$ which is exactly our bound.

Note that in the cases $g = n = 1$ and $g = 1, n = 2$ the automorphisms group of the stack differs from that of the moduli space. This is particularly evident for $\overline{\mathcal{M}}_{1,1}$, it is well known that $\overline{\mathcal{M}}_{1,1} \cong \mathbb{P}^1$ and $\overline{\mathcal{M}}_{1,1} \cong \mathbb{P}(4, 6)$. Clearly $\mathbb{P}^1 \cong \mathbb{P}(4, 6)$ as varieties, however they are not isomorphic as stacks, indeed $\mathbb{P}(4, 6)$ has two stacky points with stabilizers \mathbb{Z}_4 and \mathbb{Z}_6 . These two points are fixed by any automorphism of $\mathbb{P}(4, 6)$ while they are indistinguishable from any other point on the coarse moduli space $\overline{\mathcal{M}}_{1,1}$.

The proof of the main Theorem is essentially divided into two parts: the cases $2g - 2 + n \geq 3$ and $2g - 2 + n < 3$.

When $2g - 2 + n \geq 3$ the main tool is [GKM, Theorem 0.9] in which *A. Gibney*, *S. Keel* and *I. Morrison* give an explicit description of the fibrations $\overline{\mathcal{M}}_{g,n} \rightarrow X$ of $\overline{\mathcal{M}}_{g,n}$ on a projective variety X in the case $g \geq 1$. This result, combined with the triviality of the automorphism group of the generic curve of genus $g \geq 3$, let us to prove that the automorphisms group of $\overline{\mathcal{M}}_{g,1}$ is trivial for any $g \geq 3$. Since every genus 2 curve is hyperelliptic and has a non trivial automorphism: the hyperelliptic involution, the argument used in the case $g \geq 3$ completely fails. So we adopt a different strategy: first we prove that any automorphism of $\overline{\mathcal{M}}_{2,1}$ preserves the boundary and then we apply a famous theorem of *H. L. Royden* [Mok, Theorem 6.1] to conclude that $\mathrm{Aut}(\overline{\mathcal{M}}_{2,1})$ is trivial.

Then, applying [GKM, Theorem 0.9] we construct a morphism of groups between $\mathrm{Aut}(\overline{\mathcal{M}}_{g,n})$ and S_n . Finally we generalize *Bruno* and *Mella*'s result proving that $\mathrm{Aut}(\overline{\mathcal{M}}_{g,n})$ is indeed isomorphic to S_n when $2g - 2 + n \geq 3$.

When $2g - 2 + n < 3$ a case by case analysis is needed. In particular the case $g = 1, n = 2$ requires an explicit description of the moduli space $\overline{\mathcal{M}}_{1,2}$. Carefully analyzing the geometry of this surface we prove that $\overline{\mathcal{M}}_{1,2}$ is isomorphic to a weighted blow up of $\mathbb{P}(1, 2, 3)$ in the point $[1 : 0 : 0]$, in particular $\overline{\mathcal{M}}_{1,2}$ is toric. From this we derive that $\text{Aut}(\overline{\mathcal{M}}_{1,2})$ is isomorphic to $(\mathbb{C}^*)^2$.

Finally we consider the moduli stack $\overline{\mathcal{M}}_{g,n}$. The canonical map $\overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ induces a morphism of groups $\text{Aut}(\overline{\mathcal{M}}_{g,n}) \rightarrow \text{Aut}(\overline{\mathcal{M}}_{g,n})$. Since this morphism is injective as soon as the general n -pointed genus g curve is automorphisms free we easily derive that the automorphisms group of the stack $\overline{\mathcal{M}}_{g,n}$ is isomorphic to S_n if $2g - 2 + n \geq 3$. Then we show that $\text{Aut}(\overline{\mathcal{M}}_{1,2})$ is trivial using the fact that the canonical divisor of $\overline{\mathcal{M}}_{1,2}$ is a multiple of a boundary divisor.

This paper is organized as follows: in Section 1 we recall some basic facts about the moduli space $\overline{\mathcal{M}}_{g,n}$ and the moduli stack $\overline{\mathcal{M}}_{g,n}$, furthermore we prove some preliminary results on the fibrations of $\overline{\mathcal{M}}_{1,n}$, in Section 2 we describe explicitly the moduli space $\overline{\mathcal{M}}_{1,2}$, in Section 3 we develop the case $2g - 2 + n \geq 3$, finally in Section 4 we study the automorphisms of the stack $\overline{\mathcal{M}}_{g,n}$.

1. NOTATION AND PRELIMINARIES

We work over the field of complex numbers. Let us recall some basic facts about the moduli space $\overline{\mathcal{M}}_{g,n}$ parametrizing n -pointed stable curves of arithmetic genus g , and about the moduli stack $\overline{\mathcal{M}}_{g,n}$.

Nodal curves. The arithmetic genus g of a connected curve C is defined as $g = h^1(C, \mathcal{O}_C)$. Suppose that C has at most nodal singularities. Let $C = \bigcup_{i=1}^{\gamma} C_i$ be the irreducible components decomposition of C , and set $\delta := \sharp \text{Sing}(C)$. Let

$$\nu : \overline{C} = \bigsqcup_{i=1}^{\gamma} \overline{C}_i \rightarrow C$$

be the normalization of C . The associated morphism $\mathcal{O}_C \hookrightarrow \mathcal{O}_{\overline{C}}$ on the structure sheaves yield the following sequence in cohomology

$$0 \mapsto H^0(C, \mathcal{O}_C) \rightarrow H^0(\overline{C}, \mathcal{O}_{\overline{C}}) \rightarrow \mathbb{C}^{\delta} \rightarrow H^1(C, \mathcal{O}_C) \rightarrow H^1(\overline{C}, \mathcal{O}_{\overline{C}}) \mapsto 0.$$

We get a formula for the arithmetic genus g of C

$$g = h^1(\overline{C}, \mathcal{O}_{\overline{C}}) + \delta - \gamma + 1 = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1$$

where $g_i = h^1(\overline{C}_i, \mathcal{O}_{\overline{C}_i})$ is the geometric genus of C_i .

Definition 1.1. A *stable n -pointed curve* is a complete connected curve C that has at most nodal singularities, with an ordered collection $x_1, \dots, x_n \in C$ of distinct smooth points of C , such that the $(n + 1)$ -tuple (C, x_1, \dots, x_n) has finitely many automorphisms.

This finiteness condition is equivalent to say that every rational component of the normalization of C has at least 3 points lying over singular or marked points of C . Moduli spaces of smooth algebraic curves have been defined and then compactified adding stable curves by *Deligne* and *Mumford* in [DM]. Furthermore *Deligne* and *Mumford* proved

that, if $2g - 2 + n > 0$, there exists a coarse moduli space $\overline{M}_{g,n}$ parametrizing isomorphism classes of n -pointed stable curves of arithmetic genus g , and this space is an irreducible projective variety of dimension $3g - 3 + n$.

Boundary of $\overline{M}_{g,n}$ and dual modular graphs. The points in the boundary $\partial\overline{M}_{g,n}$ of the moduli space $\overline{M}_{g,n}$ represent isomorphism classes of singular pointed stable curves. The geometry of such curves is encoded in a graph, called dual modular graph. The boundary has a stratification whose loci, called strata, parametrize curves of a certain topological type and with a fixed configuration of the marked points.

Each nodal curve has an associated graph. This allows to represent nodal curves in a very simple way and translate some issues related to nodal curves in the language of graph theory. Let C be a connected nodal curve with γ irreducible components and δ nodes. The dual graph Γ_C of C is the graph whose vertexes represent the irreducible components of C and whose edges represent nodes lying on two components.

More precisely, each irreducible component is represented by a vertex labeled by two numbers: the genus and the number of marked points of the component. An edge connecting two vertex means that the two corresponding components intersect in the node corresponding to the edge. A loop on a vertex means that the corresponding component has a self-intersection. Recently, *S. Maggiolo* and *N. Pagani* developed a software that generates all stable dual graphs for prescribed values of g, n whose detailed description can be found in [MP]. We will use this package to generate graphs needed in this paper.

We denote by Δ_{irr} the locus in $\overline{M}_{g,n}$ parametrizing irreducible nodal curves with n marked points, and by $\Delta_{i,P}$ the locus of curves with a node which divides the curve into a component of genus i containing the points indexed by P and a component of genus $g - i$ containing the remaining points.

The closures of the loci Δ_{irr} and $\Delta_{i,P}$ are the irreducible components of the boundary $\partial\overline{M}_{g,n}$, see [Mor, Proposition 1.21].

Forgetful morphisms. For any $i = 1, \dots, n$ there is a canonical forgetful morphism

$$\pi_i : \overline{M}_{g,n} \rightarrow \overline{M}_{g,n-1}$$

forgetting the i -th marked point. If $g > 2$ and $[C, x_1, \dots, \hat{x}_i, \dots, x_n] \in \overline{M}_{g,n-1}$ is a general point the fiber

$$\pi_i^{-1}([C, x_1, \dots, \hat{x}_i, \dots, x_n]) \cong C$$

is isomorphic to C and π_i plays the role of the universal curve. Note that if $n \geq 2$ the fiber $\pi_i^{-1}([C, x_1, \dots, \hat{x}_i, \dots, x_n])$ always intersects the boundary of $\overline{M}_{g,n}$, in fact the points of the fiber corresponding to marked points represent singular curves with two irreducible components: C itself and a \mathbb{P}^1 with two marked points and intersecting C in a point. In the same way for any $I \subseteq \{1, \dots, n\}$ we have a forgetful map $\pi_I : \overline{M}_{g,n} \rightarrow \overline{M}_{g,n-|I|}$. The map π_i has sections $s_{i,j} : \overline{M}_{g,n-1} \rightarrow \overline{M}_{g,n}$ defined by sending the point $[C, x_1, \dots, \hat{x}_i, \dots, x_n]$ to the isomorphism class of the n -pointed genus g curve obtained by attaching at $x_j \in C$ a \mathbb{P}^1 with two marked points labeled by x_i and x_j .

The universal curve. The moduli space $\overline{M}_{g,1}$ with the forgetful morphism $\pi : \overline{M}_{g,1} \rightarrow \overline{M}_g$ at first glance seems to play the role of the universal curve over \overline{M}_g . However, on closer examination one realizes that $\pi^{-1}([C]) \cong C$ if and only if $[C] \in \overline{M}_g^0$ the locus of automorphisms-free curves. It is well known that the set-theoretic fiber of $\pi : \overline{M}_{g,1} \rightarrow \overline{M}_g$ over $[C] \in \overline{M}_g$ is the quotient $C/\text{Aut}(C)$. For example over an open subset of \overline{M}_2 the fibration $\pi : \overline{M}_{2,1} \rightarrow \overline{M}_2$ is a \mathbb{P}^1 -bundle and this is true even scheme-theoretically.

Remark 1.2. The situation is different if instead of considering the moduli space $\overline{M}_{g,1}$ we consider the Deligne-Mumford moduli stack $\overline{\mathcal{M}}_{g,1}$. In fact, in this case the fiber $\pi^{-1}([C])$ is isomorphic to C and via the morphism $\pi : \overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_g$ the stack $\overline{\mathcal{M}}_{g,1}$ plays the role of the universal curve over $\overline{\mathcal{M}}_g$.

Divisor classes on $\overline{\mathcal{M}}_{g,n}$. Let us briefly recall the definitions of classes λ and ψ_i on $\overline{\mathcal{M}}_{g,n}$. Consider the forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ forgetting one of the marked points and its sections $\sigma_1, \dots, \sigma_n : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf of the morphism π . The Hodge class is defined as

$$\lambda := c_1(\pi_*(\omega_\pi)).$$

The classes ψ_i are defined as

$$\psi_i := \sigma_i^*(c_1(\omega_\pi))$$

for any $i = 1, \dots, n$. Finally we denote by δ_{irr} and $\delta_{i,P}$ the boundary classes on $\overline{\mathcal{M}}_{g,n}$.

Cyclic quotient singularities. Any cyclic quotient singularity is of the form \mathbb{A}^n/μ_r , where μ_r is the group of r -roots of unit. The action $\mu_r \curvearrowright \mathbb{A}^n$ can be diagonalized, and then written in the form

$$\mu_r \times \mathbb{A}^n \rightarrow \mathbb{A}^n, (\epsilon, x_1, \dots, x_n) \mapsto (\epsilon^{a_1} x_1, \dots, \epsilon^{a_n} x_n),$$

for some $a_1, \dots, a_n \in \mathbb{Z}/\mathbb{Z}_r$. The singularity is thus determined by the numbers r, a_1, \dots, a_n . Following the notation set by *M. Reid* in [Re], we denote by $\frac{1}{r}(a_1, \dots, a_n)$ this type of singularity.

Fibrations of $\overline{\mathcal{M}}_{g,n}$. The following result by *A. Gibney, S. Keel* and *I. Morrison* gives an explicit description of the fibrations $\overline{\mathcal{M}}_{g,n} \rightarrow X$ of $\overline{\mathcal{M}}_{g,n}$ on a projective variety X in the case $g \geq 1$. We denote by N the set $\{1, \dots, n\}$ of the markings, if $S \subset N$ then S^c denotes its complement.

Theorem 1.3. (*Gibney - Keel - Morrison*) *Let $D \in \text{Pic}(\overline{\mathcal{M}}_{g,n})$ be a nef divisor.*

- *If $g \geq 2$ either D is the pull-back of a nef divisor on $\overline{\mathcal{M}}_{g,n-1}$ via one of the forgetful morphisms or D is big and the exceptional locus of D is contained in $\partial\overline{\mathcal{M}}_{g,n}$.*
- *If $g = 1$ either D is the tensor product of pull-backs of nef divisors on $\overline{\mathcal{M}}_{1,S}$ and $\overline{\mathcal{M}}_{1,S^c}$ via the tautological projection for some subset $S \subseteq N$ or D is big and the exceptional locus of D is contained in $\partial\overline{\mathcal{M}}_{g,n}$.*

The above theorem will be crucial to determine the automorphisms group of $\overline{\mathcal{M}}_{g,n}$, and can be found in [GKM, Theorem 0.9]. An immediate consequence of 1.3 is that for $g \geq 2$ any fibration of $\overline{\mathcal{M}}_{g,n}$ to a projective variety factors through a projection to some $\overline{\mathcal{M}}_{g,i}$ with $i < n$, while $\overline{\mathcal{M}}_g$ has no non-trivial fibrations. This last fact had already been shown by *A. Gibney* in her Ph.D. Thesis [G].

Such a clear description of the fibrations of $\overline{M}_{g,n}$ is no longer true for $g = 1$, an explicit counterexample to this fact was given by *R. Pandharipande* and can be found in [BM2, Example A.2], see also [Pa] for similar constructions. However, if we consider the fibrations of the type

$$\overline{M}_{1,n} \xrightarrow{\varphi} \overline{M}_{1,n} \xrightarrow{\pi_i} \overline{M}_{1,n-1}$$

where φ is an automorphism of $\overline{M}_{1,n}$, thanks to the second part of Theorem 1.3 we can prove the following lemma.

Lemma 1.4. *Let φ be an automorphism of $\overline{M}_{1,n}$. Any fibration of the type $\pi_i \circ \varphi$ factorizes through a forgetful morphism $\pi_j : \overline{M}_{1,n} \rightarrow \overline{M}_{1,n-1}$.*

Proof. By the second part of Theorem 1.3 the fibration $\pi_i \circ \varphi$ factorizes through a product of forgetful morphisms $\pi_{S^c} \times \pi_S : \overline{M}_{1,n} \rightarrow \overline{M}_{1,S} \times_{\overline{M}_{1,1}} \overline{M}_{1,S^c}$ and we have a commutative diagram

$$\begin{array}{ccc} \overline{M}_{1,n} & \xrightarrow{\varphi} & \overline{M}_{1,n} \\ \pi_{S^c} \times \pi_S \downarrow & & \downarrow \pi_i \\ \overline{M}_{1,S} \times_{\overline{M}_{1,1}} \overline{M}_{1,S^c} & \xrightarrow{\overline{\varphi}} & \overline{M}_{1,n-1} \end{array}$$

The fibers of π_i and $\pi_{S^c} \times \pi_S$ are both 1-dimensional. Furthermore φ maps the fiber of $\pi_{S^c} \times \pi_S$ over $([C, x_{a_1}, \dots, x_{a_s}], [C, x_{b_1}, \dots, x_{b_{n-s}}])$ to $\pi_i^{-1}(\overline{\varphi}([C, x_{a_1}, \dots, x_{a_s}], [C, x_{b_1}, \dots, x_{b_{n-s}}]))$. Take a point $[C, x_1, \dots, x_{n-1}] \in \overline{M}_{1,n-1}$, the fiber $\pi_i^{-1}([C, x_1, \dots, x_{n-1}])$ is mapped isomorphically to a fiber Γ of $\pi_{S^c} \times \pi_S$ which is contracted to a point $y = (\pi_{S^c} \times \pi_S)(\Gamma)$. The map

$$\overline{\psi} : \overline{M}_{1,n-1} \rightarrow \overline{M}_{1,S} \times_{\overline{M}_{1,1}} \overline{M}_{1,S^c}, [C, x_1, \dots, x_{n-1}] \mapsto y,$$

is clearly the inverse of $\overline{\varphi}$. So $\overline{\varphi}$ defines a bijective morphism between $\overline{M}_{1,S} \times_{\overline{M}_{1,1}} \overline{M}_{1,S^c}$ and $\overline{M}_{1,n-1}$, and since $\overline{M}_{1,n-1}$ is normal $\overline{\varphi}$ is an isomorphism. This forces $S = \{j\}$, $S^c = \{1, \dots, \overline{j}, \dots, n\}$. So we reduce to the commutative diagram

$$\begin{array}{ccc} \overline{M}_{1,n} & \xrightarrow{\varphi} & \overline{M}_{1,n} \\ \pi_{S^c} \times \pi_j \downarrow & & \downarrow \pi_i \\ \overline{M}_{1,1} \times_{\overline{M}_{1,1}} \overline{M}_{1,n-1} & \xrightarrow{\overline{\varphi}} & \overline{M}_{1,n-1} \end{array}$$

and $\pi_i \circ \varphi$ factorizes through the forgetful morphism π_j . □

2. THE MODULI SPACE OF 2-POINTED ELLIPTIC CURVES

Let (C, p) be a nodal elliptic curve. Then there exists $(a, b) \in \mathbb{A}^2 \setminus (0, 0)$ such that (C, p) is isomorphic to $(C', [0 : 1 : 0])$, where

$$C' = Z(zy^2 - x^3 - axz^2 - bz^3) \subset \mathbb{P}^2.$$

This representation is called *Weierstrass representation* of the elliptic curve. Consider now the 4-fold

$$X := Z(zy^2 - x^3 - axz^2 - bz^3) \subset \mathbb{A}_0^3 \times \mathbb{A}_0^2.$$

There is an action of $\mathbb{C}^* \times \mathbb{C}^* \curvearrowright X$ given by

$$\mathbb{C}^* \times \mathbb{C}^* \times X \rightarrow X, ((\lambda, \xi), (x, y, z, a, b)) \mapsto (\xi \lambda^2 x, \xi \lambda^3 y, \xi z, \lambda^4 a, \lambda^6 b).$$

The moduli stack $\overline{\mathcal{M}}_{1,1}$ is the quotient stack $[\mathbb{A}^2 \setminus (0,0)/\mathbb{C}^*] \cong \mathbb{P}(4,6)$ and the moduli space $\overline{M}_{1,1}$ is the quotient $\mathbb{A}^2 \setminus (0,0)/\mathbb{C}^* \cong \mathbb{P}^1$. There are two points of $\overline{\mathcal{M}}_{1,1}$ that are stabilized by the action of μ_4 and μ_6 respectively. These are classes of curves whose Weierstrass representations can be chosen respectively as:

$$C_4 := \{y^2 z = x^3 + xz^2\} \subset \mathbb{P}^2,$$

$$C_6 := \{y^2 z = x^3 + z^3\} \subset \mathbb{P}^2.$$

Now, $\overline{\mathcal{M}}_{1,2}$ is the universal curve over $\overline{\mathcal{M}}_{1,1}$, so $\overline{\mathcal{M}}_{1,2} = [X/\mathbb{C}^* \times \mathbb{C}^*]$ and $\overline{M}_{1,2} = X/\mathbb{C}^* \times \mathbb{C}^*$. In order to determine the singularities of $\overline{M}_{1,2}$ we have to analyze carefully the action $\mathbb{C}^* \times \mathbb{C}^* \curvearrowright X$.

Since $\overline{\mathcal{M}}_{1,2}$ is a smooth Deligne-Mumford stack the coarse moduli space $\overline{M}_{1,2}$ will have finite quotient singularities at the places where the automorphisms groups jump. Let (C, p) be a elliptic curve over \mathbb{C} , it is well known that

- $|\text{Aut}(C, p)| = 2$ if $j(C) \neq 0, 1728$,
- $|\text{Aut}(C, p)| = 4$ if $j(C) = 1728$,
- $|\text{Aut}(C, p)| = 6$ if $j(C) \neq 0$.

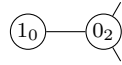
Adding a marked point will kill some automorphisms. We expect that points of type (C, p, q) with $|\text{Aut}(C, p)| = 2$ will have trivial automorphisms group. Automorphisms will jump on the points (C, p, q) with $|\text{Aut}(C, p)| = 4, 6$. To understand the behavior of the boundary $\partial \overline{M}_{1,2}$ we have to observe the following possible degenerations.

- The divisor Δ_{irr} whose general point is a curve with dual graph



and so automorphisms free.

- The divisor $\Delta_{0,2}$ whose general point is a curve with dual graph



and so with two automorphisms coming from the elliptic involution. Here we expect to get two singular points when the number of automorphisms of the elliptic curve jumps to 4 and 6.

- Two further degenerations in codimension two with the following dual graphs.



Here the automorphisms group remains of order two, so we do not expect to have singularities.

Proposition 2.1. *The moduli space $\overline{M}_{1,2}$ is a rational surface with four singular points. Two singular points lie in $M_{1,2}$, and are:*

- a singularity of type $\frac{1}{4}(2, 3)$ representing an elliptic curve of Weierstrass representation C_4 with marked points $[0 : 1 : 0]$ and $[0 : 0 : 1]$;
- a singularity of type $\frac{1}{3}(2, 4)$ representing an elliptic curve of Weierstrass representation C_6 with marked points $[0 : 1 : 0]$ and $[0 : 1 : 1]$.

The remaining two singular points lie on the boundary divisor $\Delta_{0,2}$, and are:

- a singularity of type $\frac{1}{6}(2, 4)$ representing a reducible curve whose irreducible components are an elliptic curve of type C_6 and a smooth rational curve connected by a node;
- a singularity of type $\frac{1}{4}(2, 6)$ representing a reducible curve whose irreducible components are an elliptic curve of type C_4 and a smooth rational curve connected by a node.

Proof. The rationality of $\overline{M}_{1,2}$ follows from the fact that the forgetful map $\overline{M}_{1,2} \rightarrow \overline{M}_{1,1}$ realizes $\overline{M}_{1,2}$ as a ruled surface over \mathbb{P}^1 .

To compute the singularities we study the action on X . Note that on X , $z = 0 \Rightarrow x = 0 \Rightarrow y \neq 0$. So X is covered by the charts $\{z \neq 0\}$ and $\{y \neq 0\}$.

Consider first the chart $\{z \neq 0\}$. On this chart X is given by $\{y^2 = x^3 + ax + b\}$ so $b = y^2 - x^3 - ax$. We can take (x, y, a) as coordinates, and the action of $\mathbb{C}^* \times \mathbb{C}^*$ is given by $(\lambda, x, y, a) \mapsto (\lambda^2 x, \lambda^3 y, \lambda^4 a)$. The point $(0, 0, 0)$ is stabilized by $\mathbb{C}^* \times \mathbb{C}^*$, so does not produce any singularity. Since $(2, 3) = (3, 4) = 1$ the points (x, y, a) such that $xy \neq 0$ or $ya \neq 0$ have trivial stabilizer.

If $y = 0$ the action is given by $(\lambda, x, a) \mapsto (\lambda^2 x, \lambda^4 a)$. We distinguish two cases.

- If $x = 0$ then $a \neq 0$, the stabilizer is μ_4 . So on the chart $a \neq 0$ we have a singularity of type $\frac{1}{4}(2, 3)$. Note that $x = y = 0$ implies $b = 0$. The singular point corresponds to a smooth elliptic curve of Weierstrass form C_4 and whose second marked point is $[0 : 0 : 1]$.
- If $x \neq 0$ then the stabilizer is μ_2 and on this chart we find points of type $\frac{1}{2}(1, 0)$ and these are smooth points.

If $y \neq 0$, then $\lambda^3 = 1$ and we get a singularity of type $\frac{1}{3}(2, 4)$, that is a A_2 singularity, in the point $a = x = 0$. This is a curve of type C_6 where we mark the point $[0 : 1 : 1]$. In $\overline{M}_{1,2}$ the singular point we found represents a smooth elliptic curve of Weierstrass form C_6 and whose second marked point is $[0 : 1 : 1]$.

Consider now the locus $\{z = 0\}$. We can take $y = 1$ and X is given by $\{z = x^3 + axz^2 + bz^3\}$. We are interested in a neighborhood of $x = z = 0$. Let $f(x, z, a, b) = z - x^3 - axz^2 - bz^3$ be the polynomial defining X . Since $\frac{\partial f}{\partial z}|_{z=0} \neq 0$ we can choose (x, a, b) as local coordinates. The action is given by $(\lambda, x, a, b) \mapsto (\lambda^2 x, \lambda^4 a, \lambda^6 b)$. If $x \neq 0$ the stabilizer is trivial. If $x = 0$ and $ab \neq 0$ the stabilizer is μ_2 and does not produce any singularity. We get the following two singular points.

- If $a = 0, b \neq 0$ then we have a singular point of type $\frac{1}{6}(2, 4)$. In this case we get an elliptic curve of type C_6 where we are taking the second marked point equal to the first $[0 : 1 : 0]$. So this singular point is a point on the boundary divisor $\Delta_{0,2}$ representing a reducible curve whose irreducible components are an elliptic curve of type C_6 and a smooth rational curve connected by a node.
- If $a \neq 0, b = 0$ we get a singular point of type $\frac{1}{4}(2, 6)$. We have an elliptic curve of type C_4 where the second marked point coincides with the first $[0 : 1 : 0]$. This

singular point is a point on the boundary divisor $\Delta_{0,2}$ representing a reducible curve whose irreducible components are an elliptic curve of type C_4 and a smooth rational curve connected by a node.

These two points are the only singularities on the divisor $\Delta_{0,2}$. \square

The rational Picard group of $\overline{M}_{1,2}$ is freely generated by the two boundary divisors [Be, Theorem 3.1.1]. The divisors Δ_{irr} and $\Delta_{0,2}$ are both smooth, rational curves. The boundary divisor Δ_{irr} has zero self intersection while $\Delta_{0,2}$ has negative self intersection. In [Sm] *D.I. Smyth* proves that on $\overline{M}_{1,2}$ there exists a birational morphism contracting $\Delta_{0,2}$. In the following we give a precise description of this contraction. Let us briefly recall the structure of a weighted blow up.

Remark 2.2. Let $\pi_\omega : Y \rightarrow \mathbb{C}^2$ be the weighted blow up of \mathbb{C}^2 at the origin with weight $\omega = (\omega_1, \omega_2)$,

$$Y = \{((x, y), [u : v]) \in \mathbb{C}^2 \times \mathbb{P}(\omega_1, \omega_2) \mid (x, y) \in \overline{[u : v]}\}.$$

Then Y is given by the equation $x^{\omega_1}v - y^{\omega_2}u$ in $\mathbb{C}^2 \times \mathbb{P}(\omega_1, \omega_2)$. The blow up surface Y is covered by two chart.

- On the chart $v = 1$ we have $x^{\omega_1} = y^{\omega_2}u$ and $\lambda^{\omega_2} = 1$. The action of \mathbb{C}^* is given by $\lambda \cdot (y, u) = (\lambda^{\omega_2}y, \lambda^{\omega_1}u)$, so the point $x = y = u = 0$ is a cyclic quotient singularity of type $\frac{1}{\omega_2}(\omega_1, \omega_2)$.
- On the chart $u = 1$ we have $y^{\omega_2} = x^{\omega_1}v$ and $\lambda^{\omega_1} = 1$. The action of \mathbb{C}^* is given by $\lambda \cdot (x, v) = (\lambda^{\omega_1}x, \lambda^{\omega_2}v)$, so the point $x = y = v = 0$ is a cyclic quotient singularity of type $\frac{1}{\omega_1}(\omega_1, \omega_2)$.

The singular points of Y are cyclic quotient singularities located at the exceptional divisor. Actually they coincide with the origins of the two charts.

Theorem 2.3. *The moduli space $\overline{M}_{1,2}$ is isomorphic to a weighted blow up of the weighted projective plane $\mathbb{P}(1, 2, 3)$ in its smooth point $[1 : 0 : 0]$. In particular $\overline{M}_{1,2}$ is a toric variety.*

Proof. Recall the description of $\overline{M}_{1,2}$ given at the beginning of this section. On the chart $\mathcal{U}_z := \{z \neq 0\}$ we define a morphism

$$f_{\mathcal{U}_z} : \mathcal{U}_z \rightarrow \mathbb{P}(1, 2, 3), (x, y, z, a, b) \mapsto (x, az^2, bz^3).$$

Note that the action of $\mathbb{C}^* \times \mathbb{C}^*$ on this triple is given by $(\xi\lambda^2, \xi^2\lambda^4, \xi^3\lambda^6)$, and $f_{\mathcal{U}_z}$ is indeed a well defined morphism to $\mathbb{P}(1, 2, 3)$.

On the open set $\{z \neq 0\}$ we can set $z = 1$ and ignore the action of ξ . If we forget y we can derive it up to a sign and this corresponds to the action of $\lambda = -1$.

Note that the morphism $f_{\mathcal{U}_z}$ maps the two singular point in $M_{1,2}$ we found in Proposition 2.1 in the points $[0 : 1 : 0], [0 : 0 : 1] \in \mathbb{P}(1, 2, 3)$, which are the only singularities of the weighted projective plane and of the same type of the singularities on $M_{1,2}$.

On $\mathcal{U}_y := \{y \neq 0\}$ the equation of $\overline{M}_{1,2}$ is $z = x^3 + axz^2 + bz^3$. So, as explained in the proof of Proposition 2.1 x is a local parameter near $z = 0$. We can consider the morphism

$$f_{\mathcal{U}_y}(x, y, z, a, b) = \left(1, a \left(\frac{x^2 + az^2}{1 - bz^2}\right)^2, b \left(\frac{x^2 + az^2}{1 - bz^2}\right)^3\right).$$

From this formulation it is clear that $f_{\mathcal{U}_y}$ is defined even on the locus $\{x = 0\}$ and the divisor $\Delta_{0,2} = \{x = z = 0\}$ is contracted in the smooth point $[1 : 0 : 0]$ of $\mathbb{P}(1, 2, 3)$.

On $\mathcal{U}_z \cap \mathcal{U}_y$ we have $\frac{z}{x} = \frac{x^2 + az^2}{1 - bz^2}$ and $f_{\mathcal{U}_z} = f_{\mathcal{U}_y}$, so $f_{\mathcal{U}_z}, f_{\mathcal{U}_y}$ glue to a morphism

$$f : \overline{M}_{1,2} \rightarrow \mathbb{P}(1, 2, 3).$$

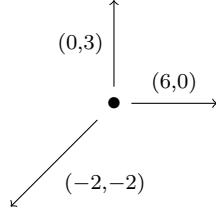
Then f is a blow up of $\mathbb{P}(1, 2, 3)$ in $[1 : 0 : 0]$ and $\Delta_{0,2}$ is the corresponding exceptional divisor. By Proposition 2.1 there are two singular points of type $\frac{1}{6}(2, 4), \frac{1}{4}(2, 6)$ on $\Delta_{0,2}$, and by Remark 2.2 the only way to obtain these two singularities is to perform a weighted blow up in $[1 : 0 : 0]$. \square

Remark 2.4. The weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ is defined by

$$\mathbb{P}(a_0, \dots, a_n) = \mathbb{P}(S),$$

where a_0, \dots, a_n are positive integers and S is the graded polynomial ring $k[x_0, \dots, x_n]$, graded by $\deg(x_i) = a_i$.

Consider the set of vectors $V = \{e_1, \dots, e_n, e_0 = -e_1 - \dots - e_n\}$ in \mathbb{R}^n and the fan whose cones are generated by proper subset of V in the lattice generated by $\frac{1}{a_i}e_i$ for $i = 0, \dots, n$. The toric variety associated to this fan is $\mathbb{P}(a_0, \dots, a_n)$. For what follows it is particularly interesting the fan of $\mathbb{P}(1, 2, 3)$:



Note that $(6, 0) + (0, 3) = 2(3, 1)$ and $(6, 0) + (-2, -2) = 2(2, -1)$. These points correspond to the two singular points of $\mathbb{P}(1, 2, 3)$. For a detailed toric description of the weighted projective space see [Ji, Section 3].

3. AUTOMORPHISMS OF $\overline{M}_{g,n}$

Our aim is to proceed by induction on n . The first step of induction is Proposition 3.5. In our argument the key fact is that the generic curve of genus $g > 2$ is automorphisms free. This is no longer true if $g = 2$ since every genus 2 curve is hyperelliptic and has a non trivial automorphism: the hyperelliptic involution. So we adopt a different strategy. First we prove that any automorphism of $\overline{M}_{2,1}$ preserves the boundary and then we apply a famous theorem of *H. L. Royden* which implies that $M_{g,n}^{un}$ (the moduli space of smooth genus g curves with unordered marked points) admits no non-trivial automorphisms or unramified correspondences for $2g - 2 + n \geq 3$, see [Mok, Theorem 6.1]. In the case $g = 1$ the following observations will be crucial.

Remark 3.1. Let $[C, x_1, x_2]$ be a two pointed elliptic curve and let x_1 be the origin of the group law on C . Let $\tau : C \rightarrow C$ be the translation mapping x_2 in x_1 , and let η be the elliptic involution. Then $\eta \circ \tau : C \rightarrow C$ is an automorphism of C switching x_1 and x_2 . Then $[C, x_1, x_2] = [C, x_2, x_1]$ and $\overline{M}_{1,2} \cong \overline{M}_{1,2}^{un}$.

Lemma 3.2. *Any automorphism of $\overline{M}_{1,2}$ and $\overline{M}_{1,3}$ preserves the divisor $\Delta_{0,2}$.*

Proof. By Theorem 2.3 the divisor $\Delta_{0,2} \subset \overline{M}_{1,2}$ is the only contractible, smooth, rational curve in $\overline{M}_{1,2}$. Then it is stabilized by any automorphism.

Let φ be an automorphism of $\overline{M}_{1,3}$ such that $\varphi(\Delta_{0,2}) \not\subseteq \Delta_{0,2}$ then composing φ with the morphism forgetting the marked point on the elliptic tail and considering the associated commutative diagram

$$\begin{array}{ccc} \overline{M}_{1,3} & \xrightarrow{\varphi} & \overline{M}_{1,3} \\ \pi_j \downarrow & & \downarrow \pi_i \\ \overline{M}_{1,2} & \xrightarrow{\overline{\varphi}} & \overline{M}_{1,2} \end{array}$$

we get an automorphism $\overline{\varphi}$ of $\overline{M}_{1,2}$ which does not preserve $\Delta_{0,2}$. \square

Lemma 3.3. [GKM, Corollary 0.12] *Any automorphism of \overline{M}_g preserves the boundary.*

Proof. Let λ be the Hodge class on \overline{M}_g . It is known that λ induces a birational morphism $f : \overline{M}_g \rightarrow X$ on a projective variety whose exceptional locus is the boundary $\partial\overline{M}_g$, see [Ru]. Assume that there exists an automorphism $\varphi : \overline{M}_g \rightarrow \overline{M}_g$ which does not preserve the boundary. Then there is a point $[C] \in \partial\overline{M}_g$ such that $\varphi([C]) = [C'] \in M_g$.

Now $f \circ \varphi$ is a birational morphism whose exceptional locus is $\varphi^{-1}(\partial\overline{M}_g)$, and by the assumption on φ we have $\varphi^{-1}(\partial\overline{M}_g) \cap M_g \neq \emptyset$. So we construct a big line bundle on \overline{M}_g whose exceptional locus is not contained in the boundary and this contradicts Theorem 1.3. \square

Proposition 3.4. *For any $g \geq 2$ the only automorphism of \overline{M}_g is the identity.*

Proof. Let φ be an automorphism of \overline{M}_g . By Lemma 3.3 φ restricts to an automorphism $\varphi|_{M_g}$ of M_g . If $g \geq 3$ by Royden's theorem [Mok, Theorem 6.1] $\varphi|_{M_g}$ is the identity, then $\varphi = Id_{\overline{M}_g}$.

If $g = 2$ the canonical divisor K_C of a smooth genus 2 curve induces a degree 2 morphism on \mathbb{P}^1 branched in 6 points. So we have a morphism

$$f : M_2 \rightarrow M_{0,6}/S_6 \cong M_{0,6}^{un}, \quad \varphi \mapsto \tilde{\varphi},$$

and since from a 6-pointed smooth rational curve we can reconstruct the corresponding genus 2 curve f is indeed an isomorphism. Then φ induces an automorphism $\tilde{\varphi}$ of $M_{0,6}^{un}$, again by [Mok, Theorem 6.1] we have $\tilde{\varphi} = Id_{M_{0,6}^{un}}$ and therefore $\varphi = Id_{\overline{M}_2}$. \square

Proposition 3.5. *For any $g \geq 2$ the only automorphism of $\overline{M}_{g,1}$ is the identity. Furthermore $\text{Aut}(\overline{M}_{1,3}) \cong S_3$.*

Proof. Let $\varphi : \overline{M}_{g,1} \rightarrow \overline{M}_{g,1}$ be an automorphism. By Theorem 1.3 the fibration

$$\pi_1 \circ \varphi : \overline{M}_{g,1} \rightarrow \overline{M}_g$$

factors through a forgetful morphism which is necessarily π_1 . We have a commutative diagram

$$\begin{array}{ccc} \overline{M}_{g,1} & \xrightarrow{\varphi} & \overline{M}_{g,1} \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ \overline{M}_g & \xrightarrow{\overline{\varphi}} & \overline{M}_g \end{array}$$

so the morphism φ maps the fiber of π_1 over $[C]$ to the fiber of π_1 over $[C'] := \overline{\varphi}([C])$. Now we distinguish two cases.

- If $g > 2$ then $\pi_1^{-1}([C])$ is a smooth genus g curve, so it is automorphisms-free. Let $[C], [C'] \in \overline{M}_g$ be two general points, then $\pi_1^{-1}([C]) \cong C$, $\pi_1^{-1}([C']) \cong C'$ and

$$\varphi|_{\pi_1^{-1}([C])} : C \rightarrow C'$$

is an isomorphism. So $C' \cong C$, $[C'] := \overline{\varphi}([C]) = [C]$ and $\overline{\varphi} = Id_{\overline{M}_g}$. We are thus reduced to a commutative triangle

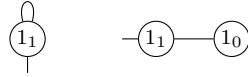
$$\begin{array}{ccc} \overline{M}_{g,1} & \xrightarrow{\varphi} & \overline{M}_{g,1} \\ & \searrow \pi_1 & \swarrow \pi_1 \\ & \overline{M}_g & \end{array}$$

and for any $[C] \in \overline{M}_g$ the restriction of φ to the fiber of π_1 defines an automorphism of the fiber. Since $g > 2$ we conclude that φ is the identity on the general fiber of π_1 so it has to be the identity on $\overline{M}_{g,1}$.

- Consider now the case $g = 2$. Let $\varphi : \overline{M}_{2,1} \rightarrow \overline{M}_{2,1}$ be an automorphism. As usual we have a commutative diagram

$$\begin{array}{ccc} \overline{M}_{2,1} & \xrightarrow{\varphi} & \overline{M}_{2,1} \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ \overline{M}_2 & \xrightarrow{\overline{\varphi}} & \overline{M}_2 \end{array}$$

The boundary of $\overline{M}_{2,1}$ has two codimension one components parametrizing curves whose dual graphs are



Similarly the boundary of \overline{M}_2 has two irreducible components parametrizing curves with dual graphs



Clearly $\pi_1(\Delta_{irr,1}) = \Delta_{irr}$ and $\pi_1(\Delta_{1,1}) = \Delta_1$. Suppose that φ maps either the class of a nodal curve or the class of the union of two elliptic curves to the class of smooth genus 2 curve then $\overline{\varphi}$ has to do the same, and this contradicts Lemma 3.3.

Then φ maps an open subset of $\partial\overline{M}_{1,2}$ to an open subset of $\partial\overline{M}_{1,2}$ and both these open sets has to intersect the irreducible components of $\partial\overline{M}_{1,2}$. Now the continuity of φ is enough to

conclude that φ preserves the boundary of $\overline{M}_{2,1}$.

Then φ restrict to an automorphism $M_{2,1} \rightarrow M_{2,1}$. By [Mok, Theorem 6.1] the only automorphism of $M_{2,1}$ is the identity. Finally $\varphi|_{M_{2,1}} = Id_{M_{2,1}}$ implies $\varphi = Id_{\overline{M}_{2,1}}$.

Consider now the case $g = 1, n = 3$. By Lemma 1.4 there exists a factorization $\pi_i \circ \varphi^{-1} = \overline{\varphi}^{-1} \circ \pi_{j_i}$, furthermore by Lemma 3.8 this factorization is unique. So we have a well defined morphism

$$\chi : \text{Aut}(\overline{M}_{1,3}) \rightarrow S_3, \varphi \mapsto \sigma_\varphi$$

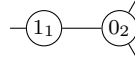
where

$$\sigma_\varphi : \{1, 2, 3\} \rightarrow \{1, 2, 3\}, i \mapsto j_i.$$

Let φ be an automorphism of $\overline{M}_{1,3}$ inducing the trivial permutation. Then we have three commutative diagrams

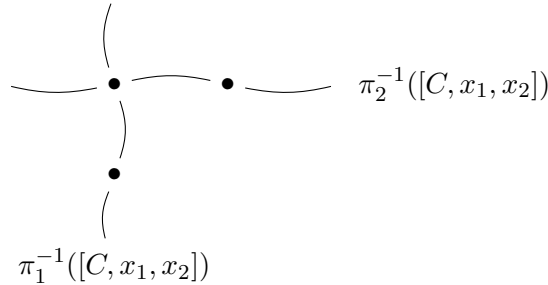
$$\begin{array}{ccc} \overline{M}_{1,3} & \xrightarrow{\varphi} & \overline{M}_{1,3} \\ \pi_i \downarrow & & \downarrow \pi_i \\ \overline{M}_{1,2} & \xrightarrow{\overline{\varphi}} & \overline{M}_{1,2} \end{array}$$

Let $[C, x_1, x_2] \in \overline{M}_{1,2}$ be a general point. The fiber $\pi_i^{-1}([C, x_1, x_2])$ intersects the boundary divisors $\Delta_{0,2} \subset \overline{M}_{1,3}$ in two points corresponding to curves with the following dual graph



The two points in $\pi_i^{-1}([C, x_1, x_2]) \cap \Delta_{0,2}$ can be identified with x_1, x_2 . Now let $[C', x'_1, x'_2]$ be the image of $[C, x_1, x_2]$ via $\overline{\varphi}$. Similarly $\pi_i^{-1}([C', x'_1, x'_2]) \cap \Delta_{0,2} = \{x'_1, x'_2\}$. By Lemma 3.2 we have $\varphi(\pi_i^{-1}([C, x_1, x_2]) \cap \Delta_{0,2}) = \pi_i^{-1}([C', x'_1, x'_2]) \cap \Delta_{0,2}$ and by Remark 3.1 $[C', x'_1, x'_2] = [C, x_1, x_2]$ and $\overline{\varphi}$ has to be identity.

So φ restrict to an automorphism of the elliptic curve $\pi_1^{-1}([C, x_1, x_2]) \cong C$ mapping the set $\{x_1, x_2\}$ into itself. On the other hand φ restricts to an automorphism of the elliptic curve $\pi_2^{-1}([C, x_1, x_2]) \cong C$ with the same property. Note that $\pi_2^{-1}([C, x_1, x_2]) \cap \pi_1^{-1}([C, x_1, x_2]) = \{x_1\}$. The situation is resumed in the following picture:



Combining these two facts we have that φ restricts to an automorphism of $\pi_1^{-1}([C, x_1, x_2]) \cong C$ fixing x_1 and x_2 . Since C is a general elliptic curve we have that $\varphi|_{\pi_1^{-1}([C, x_1, x_2])}$ is the identity, and since $[C, x_1, x_2] \in \overline{M}_{1,2}$ is general we conclude that $\varphi = Id_{\overline{M}_{1,3}}$. \square

The arguments used in the cases $g \geq 2$ and $g = 1, n \geq 3$ completely fail in the case $g = 1, n = 2$. However, Theorem 2.3 provides a very explicit description of $\overline{M}_{1,2}$ which

allows us to describe its automorphisms group. Since $\overline{M}_{1,2}$ is a toric surface we know that $(\mathbb{C}^*)^2 \subseteq \text{Aut}(\overline{M}_{1,2})$.

Remark 3.6. The automorphisms of $\mathbb{P}(a_0, \dots, a_n)$ are the automorphisms of the graded k -algebra $S = k[x_0, \dots, x_n]$. In particular the automorphisms of $\mathbb{P}(1, 2, 3)$ are of the form

$$\begin{aligned} x_0 &\mapsto \alpha_0 x_0, \\ x_1 &\mapsto \alpha_1 x_0^2 + \beta_1 x_1, \\ x_2 &\mapsto \alpha_2 x_0^3 + \beta_2 x_0 x_1 + \gamma_2 x_2, \end{aligned}$$

and the the automorphisms of $\mathbb{P}(1, 2, 3)$ fixing $[1 : 0 : 0]$ are of the form

$$\begin{aligned} x_0 &\mapsto \alpha_0 x_0, \\ x_1 &\mapsto \beta_1 x_1, \\ x_2 &\mapsto \beta_2 x_0 x_1 + \gamma_2 x_2, \end{aligned}$$

with $\alpha_0, \beta_1, \gamma_2 \in k^*$ and $\beta_2 \in k$. The composition law in this group is given by

$$(\alpha_0, \beta_1, \beta_2, \gamma_2) * (\alpha'_0, \beta'_1, \beta'_2, \gamma'_2) = (\alpha_0 \alpha'_0, \beta_1 \beta'_1, \alpha_0 \beta_1 \beta'_2 + \beta_2 \gamma'_2, \gamma_2 \gamma'_2).$$

This remark highlights why the automorphisms of the coarse moduli space $\overline{M}_{g,n}$ in general should be different from the automorphisms of the stack $\overline{\mathcal{M}}_{g,n}$. It is well known that $\overline{M}_{1,1} \cong \mathbb{P}^1$ and $\overline{\mathcal{M}}_{1,1} \cong \mathbb{P}(4, 6)$. Clearly $\mathbb{P}^1 \cong \mathbb{P}(4, 6)$ as varieties, however they are not isomorphic as stacks, indeed $\mathbb{P}(4, 6)$ has two stacky points with stabilizers \mathbb{Z}_4 and \mathbb{Z}_6 . These two points are fixed by any automorphism of $\mathbb{P}(4, 6)$ while they are indistinguishable from any other point on the coarse moduli space $\overline{M}_{1,1}$. By the previous description the automorphisms of $\overline{\mathcal{M}}_{1,1} \cong \mathbb{P}(4, 6)$ are of the form

$$\begin{aligned} x_0 &\mapsto \alpha_0 x_0, \\ x_1 &\mapsto \beta_1 x_1, \end{aligned}$$

with $\alpha_0, \alpha_1 \in k^*$.

Proposition 3.7. *The automorphisms group of $\overline{M}_{1,2}$ is isomorphic to $(\mathbb{C}^*)^2$.*

Proof. By Theorem 2.3 $\overline{M}_{1,2}$ is a weighted blow up of $\mathbb{P}(1, 2, 3)$ in $[1 : 0 : 0]$. Let φ be an automorphism of $\overline{M}_{1,2}$. Then we have a commutative diagram

$$\begin{array}{ccc} \overline{M}_{1,2} & \xrightarrow{\varphi} & \overline{M}_{1,2} \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ \overline{M}_{1,1} & \xrightarrow{\tilde{\varphi}} & \overline{M}_{1,1} \end{array}$$

and φ has to map fibers of π_1 on fibers of π_1 . Let $f : \overline{M}_{1,2} \rightarrow \mathbb{P}(1, 2, 3)$ be the contraction described in Theorem 2.3. Let $p_4, p_6 \in \Delta_{0,2}$ be the two singular points on the exceptional divisor, and let $q_4, q_6 \in M_{1,2}$ be the other two singular points. Since $\Delta_{0,2}$ is the only rational contractible curve in $\overline{M}_{1,2}$ it has to be stabilized by φ , furthermore $\varphi(p_4) = p_4$ and $\varphi(p_6) = p_6$. Let F_6 be the fiber of π_1 trough p_6, q_6 and let F_4 be the fiber of π_1 trough p_4, q_4 . Since $\varphi(q_4) = q_4$ and $\varphi(q_6) = q_6$ we get $\varphi(F_4) = F_4$ and $\varphi(F_6) = F_6$.

We denote by $L_6 := f(F_6), L_4 := f(F_4)$ the images via f of F_6 and F_4 respectively. The automorphism φ induces via f an automorphism $\tilde{\varphi}$ of $\mathbb{P}(1, 2, 3)$ fixing $[1 : 0 : 0]$ and stabilizing L_6, L_4 . Let G be the group

$$G := \{g \in \text{Aut}(\mathbb{P}(1, 2, 3)) \mid g([1 : 0 : 0]) = [1 : 0 : 0], g(L_4) = L_4, g(L_6) = L_6\},$$

and consider the morphism of groups

$$\chi : \text{Aut}(\overline{M}_{1,2}) \rightarrow G, \varphi \mapsto \tilde{\varphi}.$$

Clearly χ is injective.

Let x_0, x_1, x_2 be the coordinates on $\mathbb{P}(1, 2, 3)$. Note that the fiber F_6 corresponding to the Weierstrass curve C_6 and the fiber F_4 corresponding to the Weierstrass curve C_4 are mapped by f in the curves $L_6 = \{x_1 = 0\}$ and $L_4 = \{x_2 = 0\}$. By Remark 3.6 the automorphisms of $\mathbb{P}(1, 2, 3)$ fixing $[1 : 0 : 0]$ are of the form

$$\begin{aligned} x_0 &\mapsto \alpha_0 x_0, \\ x_1 &\mapsto \beta_1 x_1, \\ x_2 &\mapsto \beta_2 x_0 x_1 + \gamma_2 x_2, \end{aligned}$$

and forcing an automorphism to stabilize L_4 and L_6 gives $\beta_2 = 0$. Then the automorphisms in G are of the form

$$\begin{aligned} x_0 &\mapsto \alpha_0 x_0, \\ x_1 &\mapsto \beta_1 x_1, \\ x_2 &\mapsto \gamma_2 x_2, \end{aligned}$$

where $\alpha_0, \beta_1, \gamma_2 \in \mathbb{C}^*$, so $G \cong (\mathbb{C}^*)^2$. The automorphism $\tilde{\varphi}(x_0, x_1, x_2) = (\alpha_0 x_0, \beta_1 x_1, \gamma_2 x_2)$ is $\chi(\varphi)$ where φ is the automorphism of $\overline{M}_{1,2}$ acting as $\varphi(x, y, a, b) = (\alpha_0 x, \beta_1 a, \gamma_2 b)$. Consider the fibration $\overline{M}_{1,2} \rightarrow \overline{M}_{1,1}$. The automorphism φ acts on the couple (a, b) as an automorphism of $\overline{M}_{1,1} \cong \mathbb{P}^1$ and multiplying by α_0 on the fibers. So χ is surjective. \square

In order to proceed by induction on n we need the following lemma.

Lemma 3.8. *Let $\varphi : \overline{M}_{g,n} \rightarrow \overline{M}_{g,n}$ be an automorphism. For any $j = 1, \dots, n$ there exists a commutative diagram*

$$\begin{array}{ccc} \overline{M}_{g,n} & \xrightarrow{\varphi} & \overline{M}_{g,n} \\ \pi_i \downarrow & & \downarrow \pi_j \\ \overline{M}_{g,n-1} & \xrightarrow{\overline{\varphi}} & \overline{M}_{g,n-1} \end{array}$$

- The morphism $\overline{\varphi}$ is an automorphism of $\overline{M}_{g,n-1}$;
- the factorization of $\pi_j \circ \varphi$ is unique for any $j = 1, \dots, n$.

Proof. The existence of such a diagram is ensured by Theorem 1.3 and Lemma 1.4. Let $[C, x_1, \dots, x_{n-1}] \in \overline{M}_{g,n-1}$ be a point, the automorphism φ^{-1} maps isomorphically the fiber of π_j over $[C, x_1, \dots, x_{n-1}]$ to a fiber F of π_i , so $\pi_i(F) = [C', x'_1, \dots, x'_{n-1}]$ is a point. Define $\overline{\psi} : \overline{M}_{g,n-1} \rightarrow \overline{M}_{g,n-1}$ as $\overline{\psi}([C, x_1, \dots, x_{n-1}]) = [C', x'_1, \dots, x'_{n-1}]$. Clearly $\overline{\psi}$ is the inverse of $\overline{\varphi}$.

Suppose that $\pi_j \circ \varphi$ admits two factorizations $\overline{\varphi}_1 \circ \pi_i$ and $\overline{\varphi}_2 \circ \pi_h$. Then the equality $\overline{\varphi}_1 \circ \pi_i([C, x_1, \dots, x_n]) = \overline{\varphi}_2 \circ \pi_h([C, x_1, \dots, x_n])$ for any $[C, x_1, \dots, x_n] \in \overline{M}_{g,n}$ implies

$$\overline{\varphi}_1([C, y_1, \dots, y_{n-1}]) = \overline{\varphi}_2([C, y_1, \dots, y_{n-1}])$$

for any $[C, y_1, \dots, y_{n-1}] \in \overline{M}_{g,n-1}$. Now $\overline{\varphi}_1 = \overline{\varphi}_2$ implies $\overline{\varphi}_1 \circ \pi_i = \overline{\varphi}_1 \circ \pi_h$ and since $\overline{\varphi}_1$ is an isomorphism we have $\pi_i = \pi_h$. \square

At this point we can prove the general theorem by induction on n .

Theorem 3.9. *The automorphisms group of $\overline{M}_{g,n}$ is isomorphic to the symmetric group on n elements S_n*

$$\text{Aut}(\overline{M}_{g,n}) \cong S_n$$

for any g, n such that $2g - 2 + n \geq 3$.

Proof. Proposition 3.5 gives the cases $g \geq 2, n = 1$ and $g = 1, n = 3$. We proceed by induction on n . Let φ be an automorphism of $\overline{M}_{g,n}$, consider the composition $\pi_i \circ \varphi^{-1}$. By Theorem 1.3 there exists a factorization $\pi_i \circ \varphi^{-1} = \overline{\varphi^{-1}} \circ \pi_{j_i}$, furthermore by Lemma 3.8 this factorization is unique. So we have a well defined map

$$\chi : \text{Aut}(\overline{M}_{g,n}) \rightarrow S_n, \varphi \mapsto \sigma_\varphi$$

where

$$\sigma_\varphi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}, i \mapsto j_i.$$

In order to prove that σ_φ is actually a permutation we prove that it is injective. Suppose to have $\sigma_\varphi(i) = j_i = \sigma_\varphi(h)$. This means that φ^{-1} defines an isomorphism between the fibers of π_{j_i} and π_i , but also between the fibers of π_{j_i} and π_h . This forces $\pi_i = \pi_h$.

We now prove that the map χ is a morphism of groups. Let $\varphi, \psi \in \overline{M}_{g,n}$ be two automorphisms. The fibration $\pi_i \circ \psi^{-1}$ factorizes through π_{j_i} and similarly $\pi_{j_i} \circ \varphi^{-1}$ factorizes through π_{h_i} . By uniqueness of the factorization $\pi_i \circ (\psi^{-1} \circ \varphi^{-1})$ factorizes through π_{h_i} also. The situation is resumed in the following commutative diagram

$$\begin{array}{ccccc} \overline{M}_{g,n} & \xrightarrow{\varphi^{-1}} & \overline{M}_{g,n} & \xrightarrow{\psi^{-1}} & \overline{M}_{g,n} \\ \pi_{h_i} \downarrow & & \downarrow \pi_{j_i} & & \downarrow \pi_i \\ \overline{M}_{g,n-1} & \xrightarrow{\varphi^{-1}} & \overline{M}_{g,n-1} & \xrightarrow{\psi^{-1}} & \overline{M}_{g,n-1} \\ & & \searrow & \swarrow & \\ & & & & \overline{(\varphi \circ \psi)^{-1}} \end{array}$$

This means that $\sigma_\psi(i) = j_i$, $\sigma_\varphi(j_i) = h_i$ and $\sigma_{\varphi \circ \psi}(i) = h_i$. Then $\sigma_{\varphi \circ \psi}(i) = \sigma_\varphi(j_i) = \sigma_\varphi(\sigma_\psi(i))$, that is $\chi(\varphi \circ \psi) = \chi(\varphi) \circ \chi(\psi)$.

Since any permutation of the marked points induces an automorphism of $\overline{M}_{g,n}$ the morphism χ is surjective. Now we compute its kernel.

Let $\varphi \in \text{Aut}(\overline{M}_{g,n})$ be an automorphism such that $\chi(\varphi)$ is the identity, that is for any $i = 1, \dots, n$ the fibration $\pi_i \circ \varphi$ factors through π_i and we have n commutative diagrams

$$\begin{array}{ccc} \overline{M}_{g,n} & \xrightarrow{\varphi} & \overline{M}_{g,n} \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ \overline{M}_{g,n-1} & \xrightarrow{\overline{\varphi}_1} & \overline{M}_{g,n-1} \end{array} \quad \dots \quad \begin{array}{ccc} \overline{M}_{g,n} & \xrightarrow{\varphi} & \overline{M}_{g,n} \\ \pi_n \downarrow & & \downarrow \pi_n \\ \overline{M}_{g,n-1} & \xrightarrow{\overline{\varphi}_n} & \overline{M}_{g,n-1} \end{array}$$

By Lemma 3.8 the morphisms $\overline{\varphi}_i$ are automorphisms of $\overline{M}_{g,n-1}$ and by induction hypothesis $\overline{\varphi}_1, \dots, \overline{\varphi}_n$ act on $\overline{M}_{g,n-1}$ as permutations.

The action of $\overline{\varphi}_i$ on the marked points $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ has to lift to the same automorphism φ for any $i = 1, \dots, n$. So the actions of $\overline{\varphi}_1, \dots, \overline{\varphi}_n$ have to be compatible and this implies $\overline{\varphi}_i = \text{Id}_{\overline{M}_{g,n-1}}$ for any $i = 1, \dots, n$. We distinguish two cases.

- Assume $g \geq 3$. It is enough to observe that φ restricts to an automorphism of the fibers of π_1 . Then φ restricts to the identity on the general fiber of π_1 , so $\varphi = \text{Id}_{\overline{M}_{g,n}}$.

- Assume $g = 1, 2$. Note that φ restricts to an automorphism of the fibers of π_1 and π_2 . So φ defines an automorphism of the fiber of π_1 with at least two fixed points in the case $g = 1, n \geq 3$ and one fixed point in the case $g = 2, n \geq 2$. Since the general 2-pointed genus 1 curve and the general 1-pointed genus 2 curves have no non trivial automorphisms we conclude as before that φ restricts to the identity on the general fiber of π_1 , so $\varphi = Id_{\overline{M}_{g,n}}$.

This proves that χ is injective and defines an isomorphism between $\text{Aut}(\overline{M}_{g,n})$ and S_n . \square

We want to use the techniques developed in this section to recover [BM2, Theorem 4.3]. The moduli spaces $\overline{M}_{0,4}$ is isomorphic to the projective line \mathbb{P}^1 while $\overline{M}_{0,5}$ is the blow-up of \mathbb{P}^2 in four points in general position. The following is well known but we want to give a proof following the argument used in Proposition 3.5.

Proposition 3.10. *The automorphisms group of $\overline{M}_{0,5}$ is isomorphic to S_5 .*

Proof. It is well known that any fibration $\overline{M}_{0,5} \rightarrow \overline{M}_{0,4}$ factorizes through a forgetful morphism, see for instance [BM2]. This yields a surjective morphism of groups

$$\chi : \text{Aut}(\overline{M}_{0,5}) \rightarrow S_5$$

exactly as in Theorem 3.9. Let φ be an automorphism of $\overline{M}_{0,5}$ inducing the trivial permutation. Then we get five commutative diagrams

$$\begin{array}{ccc} \overline{M}_{0,5} & \xrightarrow{\varphi} & \overline{M}_{0,5} \\ \pi_i \downarrow & & \downarrow \pi_i \\ \overline{M}_{0,4} & \xrightarrow{\overline{\varphi}_i} & \overline{M}_{0,4} \end{array}$$

for $i = 1, \dots, 5$. The fiber of π_i on $[C, x_1, \dots, x_4] \in \overline{M}_{0,4}$ intersects the boundary $\partial\overline{M}_{0,4}$ in four points corresponding to x_1, \dots, x_4 . Consider $[C', x'_1, \dots, x'_4] := \overline{\varphi}_i|_{[C, x_1, \dots, x_4]}([C, x_1, \dots, x_4])$. The points in $\pi_i^{-1}([C, x_1, \dots, x_4]) \cap \partial\overline{M}_{0,4}$ and in $\pi_i^{-1}([C', x'_1, \dots, x'_4]) \cap \partial\overline{M}_{0,4}$ lie on (-1) -curves, so the automorphism φ maps the fiber of π_i over $[C, x_1, \dots, x_4]$ to the fiber of π_i over $[C', x'_1, \dots, x'_4]$ sending the set $\{x_1, \dots, x_4\}$ to the set $\{x'_1, \dots, x'_4\}$. Then $\overline{\varphi}_1, \dots, \overline{\varphi}_5$ act as permutations of the marking and since they come from the same automorphism φ they have to be compatible. This forces $\overline{\varphi}_1 = \dots = \overline{\varphi}_5 = Id_{\overline{M}_{0,4}}$.

Let $[C, x_1, \dots, x_4] \in \overline{M}_{0,4}$ be a general point. The automorphism φ restricts to an automorphism of the fiber $\pi_1^{-1}([C, x_1, \dots, x_4]) \cong \mathbb{P}^1$ stabilizing the subscheme $\{x_1, \dots, x_4\} \subset \pi_1^{-1}([C, x_1, \dots, x_4])$. Since x_1, \dots, x_4 are general points of C they have a cross-ratio different from the cross-ratio of each permutation. This means that $\varphi|_C$ is an automorphism of \mathbb{P}^1 fixing four points. So φ restricts to the identity on the general fiber of π_1 and this forces $\varphi = Id_{\overline{M}_{0,5}}$. \square

Remark 3.11. The moduli space $\overline{M}_{0,5}$ is isomorphic to a Del Pezzo surface of degree 5, by Proposition 3.10 we recover that the automorphisms group of such a surface is S_5 . For a direct proof of this classical fact which does not use the theory of moduli spaces see [DI, Section 3].

Now with the same argument of Theorem 3.9 we can prove the following:

Theorem 3.12. *The automorphisms group of $\overline{M}_{0,n}$ is isomorphic to the symmetric group on n elements S_n*

$$\mathrm{Aut}(\overline{M}_{0,n}) \cong S_n$$

for any $n \geq 5$.

Proof. The step zero of the induction is Proposition 3.10. As usual we have a surjective morphism of groups

$$\chi : \overline{M}_{0,n} \rightarrow S_n.$$

Proceeding as in the proof of Theorem 3.9 we get that an automorphism φ inducing the trivial permutation has to restrict to an automorphism of the fiber of $\pi_i : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n-1}$ fixing $k \geq 4$ points. So it has to be the identity on the general fiber of π_i , and therefore also on $\overline{M}_{0,n}$. \square

In [GKM, Corollary 0.12] *Gibney, Keel* and *Morrison* proved that any automorphism of \overline{M}_g must preserve the boundary.

From Theorem 3.9 follows immediately that the boundary of $\overline{M}_{g,n}$ has a good behavior under the action of $\mathrm{Aut}(\overline{M}_{g,n})$. The result is even stronger than the preservation of the boundary.

Corollary 3.13. *If $2g - 2 + n \geq 3$ any automorphism of $\overline{M}_{g,n}$ must preserve all strata of the boundary.*

Proof. Since any automorphism is a permutation the class of a pointed curve $[C, x_1, \dots, x_n]$ is mapped by an automorphism in a class $[C', x'_1, \dots, x'_n]$ representing a pointed curve of the same topological type of the pointed curve C . \square

4. AUTOMORPHISMS OF $\overline{\mathcal{M}}_{g,n}$

Let \mathcal{X} be an algebraic stack over \mathbb{C} . A coarse moduli space for \mathcal{X} over \mathbb{C} is a morphism $\pi : \mathcal{X} \rightarrow X$, where X is an algebraic space over \mathbb{C} such that

- the morphism π is universal for morphisms to algebraic spaces,
- π induces a bijection between $|\mathcal{X}|$ and the closed points of X , where $|\mathcal{X}|$ denotes the set of isomorphism classes in \mathcal{X} .

Remark 4.1. If \mathcal{X} admits a coarse moduli space $\pi : \mathcal{X} \rightarrow X$ then this is unique up to unique isomorphism.

A separated algebraic stack has a coarse moduli space which is a separated algebraic space [KM, Corollary 1.3].

Let \mathcal{X} be a separated stack admitting a scheme X as coarse moduli space $\pi : \mathcal{X} \rightarrow X$. The map π is universal for morphisms in schemes, that is for any morphism $f : \mathcal{X} \rightarrow Y$, with Y scheme, there exists a unique morphisms of schemes $g : X \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\pi} & X \\ & \searrow f & \swarrow g \\ & Y & \end{array}$$

commutes. Now, let $\varphi : \mathcal{X} \rightarrow \mathcal{X}$ be an automorphism of the stack \mathcal{X} , and consider $\pi \circ \varphi : \mathcal{X} \rightarrow X$. Then there exists a unique $\tilde{\varphi}$ such that the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\varphi} & \mathcal{X} \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{\tilde{\varphi}} & X \end{array}$$

commutes. By uniqueness we have $(\tilde{\varphi})^{-1} = \tilde{\varphi}^{-1}$. So $\tilde{\varphi}$ is an automorphism of X , and we get a morphism of groups

$$\mathrm{Aut}(\mathcal{X}) \rightarrow \mathrm{Aut}(X), \varphi \mapsto \tilde{\varphi}.$$

Remark 4.2. Even if \mathcal{X} is a Deligne-Mumford stack with trivial generic stabilizer the above morphism of groups is not necessarily injective. As instance in [ACV, Proposition 7.1.1] D. Abramovich, A. Corti and A. Vistoli consider a twisted curve \mathcal{C} over an algebraically closed field and its coarse moduli space C . They prove that for any node $x \in C$ the stabilizer of a geometric point of \mathcal{C} over x contributes to the automorphisms group of \mathcal{C} over C .

However since $\overline{\mathcal{M}}_{g,n}$ is a normal, Deligne-Mumford stack, as soon as its general point has trivial stabilizer, the morphism

$$\mathrm{Aut}(\overline{\mathcal{M}}_{g,n}) \rightarrow \mathrm{Aut}(\overline{M}_{g,n})$$

is injective. Our next goal is to prove this last statement.

Proposition 4.3. *The morphism of groups*

$$\mathrm{Aut}(\overline{\mathcal{M}}_{g,n}) \rightarrow \mathrm{Aut}(\overline{M}_{g,n})$$

is injective as soon as the general n -pointed genus g curve has no non trivial automorphisms.

Proof. In [FMN, Proposition A.1] take $\mathcal{X} = \mathcal{Y} = \overline{\mathcal{M}}_{g,n}$. Since we consider the case when the general n -pointed genus g curve has no non trivial automorphisms there is a dense open subscheme $U \subset \overline{M}_{g,n}$ where the canonical map $\overline{\mathcal{M}}_{g,n} \rightarrow \overline{M}_{g,n}$ is an isomorphism. Note that $\overline{\mathcal{M}}_{g,n}$ is an irreducible normal and separated Deligne-Mumford stack, so the hypothesis of [FMN, Proposition A.1] are satisfied.

Let $f : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ be an automorphism inducing the identity on the coarse moduli space $\overline{M}_{g,n}$, then there is a 2-arrow $\alpha : f|_U \Rightarrow \mathrm{Id}_U$. By [FMN, Proposition A.1] there exists a unique 2-arrow $\bar{\alpha} : f \Rightarrow \mathrm{Id}_{\overline{\mathcal{M}}_{g,n}}$ extending α . We conclude that $\bar{\alpha}$ is an isomorphism and f is isomorphic to the identity of $\overline{\mathcal{M}}_{g,n}$. \square

Theorem 4.4. *The automorphisms group of the stack $\overline{\mathcal{M}}_{g,n}$ is isomorphic to the symmetric group on n elements S_n*

$$\mathrm{Aut}(\overline{\mathcal{M}}_{g,n}) \cong S_n$$

for any g, n such that $2g - 2 + n \geq 3$. Furthermore $\mathrm{Aut}(\overline{\mathcal{M}}_g)$ is trivial for any $g \geq 2$.

Proof. For any g, n in our range the general point of $\overline{\mathcal{M}}_{g,n}$ has trivial automorphisms group. So by Proposition 4.3 the morphism of groups

$$\mathrm{Aut}(\overline{\mathcal{M}}_{g,n}) \rightarrow \mathrm{Aut}(\overline{M}_{g,n})$$

is injective. By Theorem 3.9 and [BM2, Theorem 4.3] we know that $\text{Aut}(\overline{\mathcal{M}}_{g,n}) \cong S_n$ for the values of g and n we are considering. Since any permutation of the marked points in an automorphism of $\overline{\mathcal{M}}_{g,n}$ we conclude that

$$\text{Aut}(\overline{\mathcal{M}}_{g,n}) \cong \text{Aut}(\overline{\mathcal{M}}_{g,n}) \cong S_n.$$

Since the general curve of genus $g \geq 3$ is automorphisms free the morphism

$$\text{Aut}(\overline{\mathcal{M}}_g) \rightarrow \text{Aut}(\overline{\mathcal{M}}_g)$$

is injective. We conclude by Proposition 3.4. In the case $g = 2$ consider the fiber product

$$\begin{array}{ccc} \overline{\mathcal{M}}_{2,1} \times_{\overline{\mathcal{M}}_2} \overline{\mathcal{M}}_2 & \xrightarrow{\psi} & \overline{\mathcal{M}}_{2,1} \\ \downarrow & \searrow \varphi & \downarrow \pi_1 \\ \overline{\mathcal{M}}_2 & \xrightarrow{\varphi} & \overline{\mathcal{M}}_2 \end{array}$$

where $\varphi \in \text{Aut}(\overline{\mathcal{M}}_2)$. Since φ is an automorphism ψ also is an automorphism. By the previous part of the proof we know that $\text{Aut}(\overline{\mathcal{M}}_{2,1}) \cong \text{Aut}(\overline{\mathcal{M}}_{2,1})$ is trivial. So $\psi = \text{Id}_{\overline{\mathcal{M}}_{2,1}}$ and therefore $\varphi = \text{Id}_{\overline{\mathcal{M}}_2}$. \square

As we saw in Proposition 3.7 the case $g = 1, n = 2$ is pathological from the point of view of the automorphisms. Since $\text{Aut}(\overline{\mathcal{M}}_{1,2}) \cong (\mathbb{C}^*)^2$ the injectivity of the morphism $\text{Aut}(\overline{\mathcal{M}}_{1,2}) \rightarrow \text{Aut}(\overline{\mathcal{M}}_{1,2})$ does not say to much on $\text{Aut}(\overline{\mathcal{M}}_{1,2})$. Since all the automorphisms of $\overline{\mathcal{M}}_{1,2}$ are toric we expect them to disappear on the stack. In the following proposition we prove that $\text{Aut}(\overline{\mathcal{M}}_{1,2})$ is trivial exploiting the particular form of its canonical divisor.

Proposition 4.5. *The only automorphism of the moduli stack $\overline{\mathcal{M}}_{1,2}$ is the identity.*

Proof. An application of the Grothendieck-Riemann-Roch theorem [HM, Section 3E] gives the following formula for the canonical class of $\overline{\mathcal{M}}_{1,2}$

$$K_{\overline{\mathcal{M}}_{1,2}} = 13\lambda - 2\delta + \psi \in \text{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{1,2}).$$

The Picard group $\text{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{1,2})$ is freely generated by λ and the boundary classes, furthermore the following relations hold [AC, Theorem 2.2]:

$$\delta_{irr} = 12\lambda, \quad \psi = 2\lambda + 2\delta_{0,2}.$$

We can write the canonical class in terms of the boundary divisors as

$$K_{\overline{\mathcal{M}}_{1,2}} = \frac{13}{12}\delta_{irr} - 2\delta_{irr} - 2\delta_{0,2} + \frac{2}{12}\delta_{irr} + 2\delta_{0,2} = -\frac{3}{4}\delta_{irr}.$$

Note that δ_{irr} is a fiber of the forgetful morphism $\pi_1 : \overline{\mathcal{M}}_{1,2} \rightarrow \overline{\mathcal{M}}_{1,1}$. Any automorphism φ of $\overline{\mathcal{M}}_{1,2}$ preserves the canonical bundle, that is $\varphi^* K_{\overline{\mathcal{M}}_{1,2}} = K_{\overline{\mathcal{M}}_{1,2}}$ in $\text{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{1,2})$. Since $K_{\overline{\mathcal{M}}_{1,2}}$ is a multiple of the fiber δ_{irr} the fibration $\pi_1 \circ \varphi$ factorizes through π_1 (recall that by Remark 3.1 on $\overline{\mathcal{M}}_{1,2}$ the forgetful morphisms induce the same fibration). So we have the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{1,2} & \xrightarrow{\varphi} & \overline{\mathcal{M}}_{1,2} \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ \overline{\mathcal{M}}_{1,1} & \xrightarrow{\overline{\varphi}} & \overline{\mathcal{M}}_{1,1} \end{array}$$

Let $[C, p] \in \overline{\mathcal{M}}_{1,1}$ be a general point and let $[C', p'] = \overline{\varphi}([C, p])$ be its image. Then $\alpha := \varphi|_{\pi_1^{-1}([C, p])}$ defines an isomorphism between C and C' . If $q' = \alpha(p)$ then there exists an automorphism τ' of C' mapping q' to p' . So $\tau' \circ \alpha$ is an isomorphism between C and C' mapping p to p' . This means that $[C, p] = [C', p']$, $\overline{\varphi}$ is the identity and φ restricts to an automorphism of the fiber of π_1 , furthermore by Lemma 3.2 has to preserve the boundary divisor $\delta_{0,2}$. The general fiber of π_1 is a general elliptic curve, so it has only two automorphisms. Clearly both these automorphisms act trivially on $\overline{\mathcal{M}}_{1,2}$, so $\varphi = Id_{\overline{\mathcal{M}}_{1,2}}$. \square

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